ma
the
ma
tisch

cen
trum

AFDELING ZUIVERE WISKUNDE (DEPARTMENT OF PURE MATHEMATICS)

ZN 65/76 AUGUSTUS

J. DE VRIES

COSEPARATORS IN CATEGORIES OF TOPOLOGICAL TRANSFORMATION GROUPS

amsterdam 1976

stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE (DEPARTMENT OF PURE MATHEMATICS) ZN 65/76 AUGUSTUS

J. DE VRIES

COSEPARATORS IN CATEGORIES OF TOPOLOGICAL TRANSFORMATION GROUPS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 54H15, 18B99

Coseparators in categories of topological transformation groups

by

J. de Vries

ABSTRACT

The main result in this note is that the category $COMP^G$ of all compact G-spaces has a coseparator, provided G is a locally compact Hausdorff topological group. This provides a partial solution to an open question raised earlier by the author.

KEY WORDS & PHRASES: *compact topological transformation group, coseparator in a category,* G-*space,* G-*compactification.*

In certain parts of mathematics the question is of interest whether all members of a given class of objects can be embedded in one "comprehensive" object. See for example [1]. This sort of problem, when studied in a categorical context, leads to the concept of a *coseparator*. See [2; 19.6] or [3; 24.6.5].

As to the existence of "comprehensive objects" for certain classes of topological transformation groups (ttg's) we refer to [4; Chap. III]. These results are derived independently of coseparators, but nevertheless it is interesting to know which categories of ttg's have a coseparator. In [4; section 6.4] we obtained some results in this direction as consequences of a general theorem about "preservation of coseparators" by certain functors. However, the question of whether the category of all compact Hausdorff G-spaces has a coseparator was left open. In this note we give an affirmative answer for the case that G is a locally compact Hausdorff group.

In the sequel, G shall always denote a locally compact Hausdorff topological group with identity element e. Recall that *a* G-*space* is a pair $\langle X,\pi\rangle$ in which X a topological space and $\pi: G\times X \to X$ is a continuous mapping satisfying the conditions: $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for all $s, t \in G$ and $x \in X$. If $\langle X,\pi\rangle$ and $\langle Y,\sigma\rangle$ are G-spaces, then a *morphism of* G-*spaces* $f: \langle X,\pi\rangle \to \langle Y,\sigma\rangle$ is a continuous mapping $f: X \to Y$ such that $f(\pi(t,x)) = \sigma(t,f(x))$ for all $t \in G$, $x \in X$. In this way we obtain the category of all G-spaces and all morphisms of G-spaces, denoted TOP^G. If $\mathcal{B}$ is a full subcategory of *TOP*, then the corresponding full subcategory of TOP^G is denoted $\mathcal{B}^G$. As a general reference for categories of ttg's, see [4] and [5]. If X is any topological space, then $C_c(G,X)$ denotes the space of all continuous functions from G into X, endowed with the compact-open topology. If $\tilde{\rho}_X: G\times C_c(G,X) \to C_c(G,X)$ is defined by

$$\tilde{\rho}_X(t,f)(s) := f(st)$$

for $f \in C_c(G,X)$ and $t, s \in G$ (so each $\tilde{\rho}_X{}^t$ is a right-translation of functions), then $\langle C_c(G,X),\tilde{\rho}_X\rangle$ is a G-space ($\tilde{\rho}_X$ is continuous because G is locally compact). In the sequel we shall always omit the subscript X in $\tilde{\rho}_X$.

<u>PROPOSITION 1</u>. *Let $\mathcal{B}$ denote a full subcategory of TOP which has a coseparator* X *such that* $C_c(G,X)$ *is an object in* $\mathcal{B}$. *Then* $\langle C_c(G,X),\tilde{\rho}\rangle$ *is a*

coseparator in $\mathcal{B}^G$.

PROOF. Let $\langle Y,\sigma\rangle$ be any object in $\mathcal{B}^G$ and let $y_1, y_2 \in Y$, $y_1 \neq y_2$. It is sufficient to show that there exists a morphism of G-spaces $f: \langle Y,\sigma\rangle \to \langle C_c(G,X), \tilde{\rho}\rangle$ with $f(y_1) \neq f(y_2)$. Since X is een coseparator in $\mathcal{B}$ there exists a continuous function $g: Y \to X$ such that $g(y_1) \neq g(y_2)$. Define $f: Y \to C_c(G,X)$ by

$$f(y)(t) := g(\sigma(t,y)),\ y \in Y,\ t \in G.$$

It is easily checked that $f: Y \to C_c(G,X)$ is continuous. Moreover, by direct computation one can verify that $f: \langle Y,\sigma\rangle \to \langle C_c(G,X),\tilde{\rho}\rangle$ is a morphism of G-spaces. Since

$$f(y_1)(e) = g(y_1) \neq g(y_2) = f(y_2)(e)$$

we have $f(y_1) \neq f(y_2)$, as desired. □

EXAMPLES (cf. also [4; section 6.4]).

1. The indiscrete two-point space E_2 is a coseparator in *TOP*. Hence $\langle C_c(G,E_2),\tilde{\rho}\rangle$ is a coseparator in TOP^G.
2. Let F_2 denote the two-point space $\{0,1\}$ with the T_0-topology $\{\phi,\{0\},\{0,1\}\}$. Then $\langle C_c(G,F_2),\tilde{\rho}\rangle$ is a coseparator in the full subcategory of TOP^G, determined by all T_0 G-spaces.
3. The discrete two-point space D_2 is a coseparator in the full subcategory TOP_0 of all zero-dimensional Hausdorff spaces. Since $C_c(G,D_2)$ is also zero-dimensional, it follows that $C_c(G,D_2)$ is a coseparator in TOP^G.
4. The closed unit interval I is a coseparator in the category *TYCH* of all Tychonoff (= completely regular Hausdorff) spaces. Since $C_c(G,I)$ is a Tychonoff space, $\langle C_c(G,I),\tilde{\rho}\rangle$ is a coseparator in $TYCH^G$.
5. Observe that $C_c(G,I)$ is compact iff G is discrete. [If G is discrete, $C_c(G,I) = I^G$ is compact by the Tychonoff-theorem. Conversely, if $C_c(G,I)$ is compact, then $C_p(G,I)$ is compact. But $C_p(G,I)$ is dense in I^G, hence it coincides with I^G.] Consequently, *unless G is discrete, our method does not provide a coseparator for* $COMP^G$ (*COMP* is the category of compact Hausdorff spaces).

THEOREM 2. *For any locally compact Hausdorff group* G *the category* $COMP^G$ *has a coseparator.*

PROOF. In [5] it is shown that every G-space has a G-compactification (G locally compact). For the G-space $\langle C_c(G,I),\tilde{\rho}\rangle$ this means that there exists a morphism h: $\langle C_c(G,I),\tilde{\rho}\rangle \to \langle Z,S\rangle$ in TOP^G such that Z is a compact Hausdorff space and h is a topological embedding of $C_c(G,I)$ in Z. In particular, k is injective. It follows immediately from EXAMPLE 4 above and the injectivity of h, that $\langle Z,G\rangle$ is a coseparator in $TYCH^G$. Because $\langle Z,G\rangle$ is an object in $COMP^G$, it follows that it is a coseparator in $COMP^G$. . □

REMARK. It can be shown that the weight $w(Z)$ of the space Z mentioned in the above proof equals $w(G)$, the weight of G. So if G is a separable metrizable group (and, of course, locally compact) then Z is second-countable, hence compact and metrizable; therefore, in this case $\langle Z,S\rangle$ is also a coseparator for the category of all compact metrizable G-spaces. It would be of interest to find a coseparator for this category under weaker conditions on G. [For example: G sigma-compact. In this context, observe that for a sigma-compact group G, $C_c(G,I)$ is metrizable, so that $\langle C_c(G,I),\tilde{\rho}\rangle$ in a coseparator in the category of all metrizable G-spaces.] Another open question is, whether the category $COMP^G$ has *injective* coseparator.

REFERENCES

[1] BAAYEN, P.C., *Universal Morphisms*, Mathematical Centre Tracts no. 9, Mathematisch Centrum, Amsterdam, 1964.

[2] HERRLICH, H. & G.E. STRECKER, *Category Theory*, Allyn and Bacon Inc., Boston, 1973.

[3] SEMADENI, Z., *Banach Spaces of Continuous Functions*, I, PWN, Warszawa, 1971.

[4] VRIES, J. DE, *Topological Transformation Groups: a categorical approach*, Mathematical Centre Tracts no. 65, Mathematisch Centrum, Amsterdam, 1975.

[5] ______ , *Categories of topological transformation groups*, to appear in

the Proceedings of the International Conference on Categorical Topology in Mannheim, 1975.

[6] ———, *On the existence of* G-*compactifications*, submitted for publication.